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ON (n,m)-GROUPS

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Abstract. The main result of the article is the following proposition. Let $n \geq 2m$ and let (Q,A) be an (n,m)-groupoid. Then, (Q,A) is an (n,m)–group iff the following stetements hold: (i) (Q,A) is an $< 1, n-m+1 >$– and $< 1,2 >$-associative (n,m)-groupoid /or $< 1, n-m+1 >$– and $< n-m, n-m+1 >$-associative (n,m)-groupoid /; and (ii) for every $a_1^n \in Q$ there is **at least one** $x_1^m \in Q^m$ and **at least one** $y_1^m \in Q^m$ such that the following equalities hold $A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$ and $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$. [For $n=2$ and $m=1$ it is a well known characterization of a group. See, also 3.2.]

1. Introduction

Definitions 1.1. *Let $n \geq m+1$ $(n,m \in N)$ and (Q,A) be an (n,m)-groupoid $(A: Q^n \to Q^m)$. Then: (a) we say that (Q,A) is an (n,m)-***semigroup** *iff for every $i,j \in \{1,\dots,n-m+1\}, i<j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$

/: $< i,j >$-associative law /; and (b) we say that (Q,A) is an (n,m)-**group** iff (Q,A) is an (n,m)-semigroup and for every $a_1^n \in Q$ there is **exactly one** sequence x_1^m over Q and **exactly one** sequence y_1^m over Q such that the following equalities hold

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n. \text{ (See, also [6].)}$$

Remark 1.2. A notion of an (n,m)-group was introduced by G. Čupona in [5] as a generalization of the notion of a group (n-group – [1]). The paper [6] is mainly a survey on the known results for vector valued groupoids, semigroups and groups (to 1988). □

2. Auxiliary propositions

In this paper the following $< i,j >$-associative laws have the prominence:

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(1) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$,

(1L) $A(A(x_1^n), x_{n+1}^{2n-m}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-m})$, and

(1R) $A(x_1^{n-m-1} A(x_{n-m}^{2n-m-1}), x_{2n-m}) = A(x_1^{n-m}, A(x_{n-m+1}^{2n-m}))$.

Proposition 2.1. [9]. *Let $n \geq 2m$ and let (Q, A) be an (n, m)-groupoid. Further on, let the $< 1, n-m+1 >$-associative law [:(1)] holds in (Q, A) and let for every $a_1^n \in Q$ there is* **at least one** *sequence x_1^m over Q and* **at least one** *sequence y_1^m over Q such that the following equalities hold*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then, there exists mapping $^{-1}$ of the set Q^{n-m} into the set Q^m such that the following laws hold in the algebra $(Q, \{A, ^{-1}\})$

(2L) $A((a_1^{n-2m}, b_1^m)^{-1}, a_1^{n-2m}, A(b_1^m, a_1^{n-2m}, x_1^m)) = x_1^m$ and

(2R) $A(A(x_1^m, a_1^{n-2m}, b_1^m), a_1^{n-2m}, (a_1^{n-2m}, b_1^m)^{-1}) = x_1^m$.

Proposition 2.2. [9]: *Let $n > m+1$ and let (Q, A) be an (n, m)-groupoid. Also let*

(a) the (1L) [(1R)] law holds in (Q, A); and

(b) for every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$$
$$[A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m].$$

Then, (Q, A) is an (n, m)-semigroup.

Remark 2.3 For $m = 1$ propositions 2.1 and 2.2 is proved in [7].

3. Main result

Theorem 3.1. *Let $n \geq 2m$ and let (Q, A) be an (n, m)-groupoid. Then, (Q, A) is an (n, m)-group iff the following statements hold:*

(i) the laws (1) and (1L) hold the (n, m)-groupoid (Q, A) [**or** *the laws (1) and (1R) hold in (Q, A)]; and*

(ii) for every $a_1^n \in Q$ there is **at least one** *$x_1^m \in Q^m$ and* **at least one** *$y_1^m \in Q^m$ such that the following equalities hold*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Proof. *a)* $\Rightarrow$: By the definition of an (n, m)-group, immediately we conclude that the statements (i) and (ii) hold.

b) $\Leftarrow$:

b_1) For $m = 1$ " $\Leftarrow$ " is proved in [8].

b_2) Let $m > 1$. Then the following statement holds:

$1°$ $(\forall m \in N)(\forall n \in N)(m > 1 \Rightarrow (n \geq 2m \Rightarrow n > m+1))$.

Furthermore, the following statements hold:

2° For every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m$$

/: (1) from 2, (*ii*), 2.1–(2R)/; and

3° for every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds

$$A(a_1^{n-m}, x_1^m) = A(a_1^{n-m}, y_1^m) \Rightarrow x_1^m = y_1^m$$

/: (1) from 2(-(*i*)), (*ii*), 2.1–(2L)/.

By 1°, 2°, (1L) /(1R)/ and 2.2, we conclude that the following statement holds:

4° (Q, A) is an (n,m)-semigroup.

Finally, by (*ii*), 2°, 3°, 4° and 1.1, we conclude that (Q, A) is an (n,m)-group.

The direction " $\Leftarrow$ " holds therefore for $m > 1$ as well. □

Remarks 3.2. *a*) For $m = 1$ Theorem 2.1 is proved in [8]. *b*) The following proposition has been proved in [3]: An n-**semigroup** (Q, A) is an n-group iff for each $a_1^n \in Q$ there exists **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $A(a_1^{n-1}, x) = a_n$ and $A(y, a_1^{n-1}) = a_n$. This assertion has been already formulated in [2], but the proof is missing there. W.A.Dudek has pointed my attention to this fact.

4. Two propositions more

Proposition 4.1. *Let $n \geq 2m$ and let (Q, A) be an (n,m)-groupoid. Then, (Q, A) is an (n,m)-group iff the following statements hold:*

(I) *the law (1L) from 2 hold in (Q, A); and*

(II) *for every $a_1^n \in Q$ there is* **at least one** *$x_1^m \in Q^m$ and* **exactly one** *$y_1^m \in Q$ such that the following equalities hold:*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Proof. *a*) $\Rightarrow$: By the definition of an (n,m)-group, immediately we conclude that the statements (I) and (II) hold.

b) $\Leftarrow$:

b_1) For $(n,m) = (2,1)$ (Q, A) is a group.

b_2) Let $(n,m) \neq (2,1)$. Then the following statement holds:

°1 $(\forall m \in N)(\forall n \in N \backslash \{1\})((n,m) \neq (2,1) \Rightarrow (n \geq 2m \Rightarrow n > m+1))$.

Moreover, by (II), we conclude that the following statement holds:

°2 For every $u_1^m, v_1^m, a_1^{n-m} \in Q$ the following implication holds

$$A(u_1^m, a_1^{n-m}) = A(v_1^m, a_1^{n-m}) \Rightarrow u_1^m = v_1^m.$$

By °1, °2, (I) and 2.2, we conclude that the following statement holds:

°3 The law (1) from 2 holds in (Q,A).

Finally (I), °3, (II) and by 3.1, we conclude that (Q,A) is an (n,m)–group.

So, the direction " $\Leftarrow$ " holds also for $(n,m) \neq (2,1)$. □

Similarly, it is possible to prove that the following proposition holds:

Proposition 4.2. *Let $n \geq 2m$ and let (Q,A) be an (n,m)-groupoid. Then, (Q,A) is an (n,m)-group iff the following statements hold:*

$(\widehat{I})$ *The law (1R) from 2 holds in (Q,A); and*

$(\widehat{II})$ *For every $a_1^n \in Q$ there is* **exactly one** *$x_1^m \in Q^m$ and* **at least one** *$y_1^m \in Q^m$ such that the following equalities hold:*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n \text{ and } A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Remark: For $m = 1$ propositions 4.1 and 4.2 is proved in [4].

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